NOTATION

T, A, U, B, Q, matrices and vectors of the appropriate dimensions; $[\ ]^{-1}$, sign of matrix inversion $[\ ]^{t}$, sign of matrix transposition; λ, thermal conductivity; c, heat capacity; ρ, specific density; ε, ratio of semiaxes of ellipse; N, number of measurements per section; Δ, length of approximation section; σ^2, dispersion of noise.

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SOME ANALYTICAL METHODS OF SOLVING INVERSE (COEFFICIENT) PROBLEMS OF HEAT-CONDUCTION THEORY

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Some analytical methods are presented for the determination of thermophysical parameters without linearization of the heat-conduction equation. A qualitative study of the temperature fields is used.

1. The mathematical description of intense heat-transfer processes is connected with the necessity of allowing for the temperature dependence of the thermophysical parameters. For this, in the one-dimensional case, the nonlinear heat conduction is written as

$$c(T)\,\gamma(T)\,\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left[\Lambda(T)\,\frac{\partial T}{\partial x}\right]+\frac{\Gamma}{x}\,\Lambda(T)\,\frac{\partial T}{\partial x}+F(x,\ \tau,\ T). \tag{1}$$

We note that until recently insufficient attention has been paid to the mathematical side of the determination of thermophysical parameters, especially to questions of the accuracy and of the errors which are introduced. The complexity of the determination of thermophysical parameters has been aggravated by the absence of exact analytical solutions for (1). It is just these reasons (during the time which preceded the extensive use of electronic and analog computers and the consideration of questions of the correctness of the solutions of inverse problems) which forced investigators to use various approximate solutions (most often linearized ones). In this case nonlinear parameters were replaced by piecewise-linear parameters and so forth. The errors introduced in the process do not yield to analysis in general form, which prevents one from giving a reliable estimate of the accuracy of the parameters obtained, especially when they are strongly nonlinear. As an illustration we cite the following two examples. As is known, one of the methods of determining parameters often applied in engineering practice is the method of the regular regime of type I [1-2]. In this case one is confined to one (or several) terms of the series in the calculating equations for the linearized solutions of (1). The error introduced in the process (the remainder of the series) has usually been taken as the error in the determination of the parameter. In fact, there are two kinds of errors: those for direct and inverse problems [3],

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with the latter being some function of the former. A study showed that in any temperature range under consideration there is always an interval where the error in the determination of the coefficient of thermal diffusivity varies from 0 to 100% (in [3] such an interval was equated to almost two thirds of the temperature range under consideration). The ignoring of this situation has led, in our view, to considerable understating of the errors in the determination of parameters at the traditional 1-5%. The second example is connected with the erroneous concept which has become implanted in the literature concerning the use of calculating equations based on the integral mean values of the temperatures instead of the local values. This is fundamentally wrong, since the piecewise constant parameter obtained in the process ultimately gives a dependence on the integral mean temperature rather than the local values (the parameters enter into the heat-conduction equation as functions of the local values).

Methods based on the various published solutions of (1) are not universal, as a rule: They are only suitable for weakly nonlinear parameters, close to linear (see the review in [4], for example).

Methods applicable for any form of nonlinear dependence of the parameters on the temperature are proposed in the present report.

2. Henceforth we assume that the experimentally obtained temperature fields are already statistically analyzed.

At the basis of the methods proposed below lies the qualitative study of temperature fields [5]: of a steady field for an unbounded plate and of a nonsteady self-similar field for a semibounded body. Thus, in the case of a steady temperature field obtained with boundary conditions of the first kind for an unbounded plate the following is valid: If the temperature field has inflection points, then their ordinates are simultaneously extremal points of the unknown coefficient of thermal conductivity, dividing the range of variation of the temperature into intervals of monotonicity; if the temperature field is represented by a convex (concave) curve then the unknown parameter is accordingly an increasing (decreasing) function of the temperature. For a self-similar regime the presence of one inflection point on the graph of the temperature field indicates an increase (decrease) in the unknown parameter in accordance with the decrease (increase) in the temperature field [5]. The presence of two inflection points indicates an extremum of the unknown parameter whose abscissa lies between the ordinates of the inflection points [5], etc.

We note that when a self-similar solution $\theta = \theta(\xi)$ is used in an experiment it is sufficient to have values of the temperature field measured at only one point x_1 at different times τ_j: $T(x_1, \tau_j)$, which then take the form $\theta = \theta(\xi)$ with the help of a change of variables. The presence of experimental values of the temperature field at only one or two points (in the steady case) also allows one to draw certain conclusions concerning the increase (decrease) of the physical parameter and the presence of extrema [5-6].

3. Nonsteady Self-Similar Solutions. Let us consider self-similar solutions of (1) for a semibounded body ($\Gamma = 0$), assuming that the source function $F(x, \tau, T)$ can be written in self-similar form, such as

$$F(x, \tau, T) = \varphi(T)\frac{\partial T}{\partial \tau}. \tag{2}$$

We note that Eq. (2) analytically describes various phase transitions in particular (crystallization, evaporation, etc.).

The boundary conditions are written in the form

$$T(0, \tau) = T_1, \quad T(x, 0) = T_0, \quad \frac{\partial T(\infty, \tau)}{\partial x} = 0. \tag{3}$$

With the allowance for the relations

$$\theta = \frac{T - T_0}{T_1 - T_0}, \quad \frac{\Lambda(T)}{\Lambda_0} = \Lambda(\theta), \quad \frac{c(T)\gamma(T) - \varphi(T)}{c_0\gamma_0} = \Phi(\theta), \tag{4}$$

$$\xi = (4a_0\tau)^{-0.5}x, \quad a_0 = \Lambda_0 c_0^{-1}\gamma_0^{-1} \tag{5}$$

we rewrite the boundary problem (1)-(3) as

$$-\theta'[2\xi\Phi(\theta) + \Lambda'(\theta)\theta'] = \Lambda(\theta)\theta'', \tag{6}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0, \quad \theta'(\infty) = 0. \tag{7}$$

TABLE 1. Results of Calculations by Eqs. (9)-(11)

ξ_i	θ_i	$-\Delta\theta_i$	$-\theta'(\xi_i)$	$\Lambda_i(\theta_i)$	$\Lambda_i^{(1)}(\theta_i)$	$\delta_i^{(1)}$, %	$\Lambda_i^{(2)}(\theta_i)$	$\delta_i^{(2)}$, %
0	1			5	5	0	5	0
0,125	0,950	0,100	0,400	4,110	4,840	13,850	4,540	4,620
0,250	0,900							
0,375	0,825	0,150	0,600	2,940	2,970	1,010	2,920	0,685
0,500	0,750							
0,625	0,675	0,180	0,720	2,180	2,285	4,600	2,190	0,456
0,750	0,570							
0,875	0,470	0,190	0,760	1,605	1,685	4,750	1,163	1,530
1	0,380							
1,125	0,300	0,170	0,680	1,315	1,342	2,100	1,316	0,076
1,250	0,210							
1,375	0,150	0,110	0,440	1,135	1,180	3,810	1,145	0,690
1,500	0,100							
1,675	0,075	0,060	0,240	1,065	1,175	9,370	1,166	6,060
1,750	0,040							
1,875	0,025	0,020	0,096	1,020	1,120	8,900	1,100	7,270
2	0,020							
2,125	0,018	0,010	0,040	1,010	1,117	14,000	1,130	10,600
2,250	0,010							
2,375	0,005	0,010	0,040	1,005				
2,500	0							

I. The Method of Direct Integration. By integrating (6) relative to the unknown parameter we obtain

$$\Lambda(\theta) = \frac{1}{\theta'(\xi)}\left[\Lambda(\theta_*)\,\theta'(\theta_*) - 2\int_{\theta_*}^{\theta}\Phi(\theta_1)\,\xi(\theta_1)\,d\theta_1\right] \quad (0 \leqslant \theta_* \leqslant 1). \tag{8}$$

The integral in (8) is calculated approximately. We assume that $\varphi(T)$, $c(T)$, and $\gamma(T)$ are known. It is shown in [7] that the coefficients of thermal conductivity and heat capacity cannot be found simultaneously as independent functions.

Example. Using Fudzit's data [8], from Eq. (8) we find the coefficient of thermal conductivity ($\varphi \equiv 0$, $\Phi(\theta) \equiv 1$), which has the form

$$\Lambda(\theta) = (1 - 0.8\theta)^{-1}. \tag{9}$$

The integral in (8) is calculated by Simpson's rule ($\theta_* = 0$) with n = 2

$$\Lambda_*^{(1)}(\theta_i) = \frac{-\theta_i}{3\theta_i'(\xi_i)}\,[\xi(\theta_i) + 4\xi(0.5\theta_i) + 2.5] \tag{10}$$

and n = 4

$$\Lambda_*^{(2)}(\theta_i) = \frac{-\theta_i}{6\theta_i'(\xi_i)}\,[\xi(\theta_i) + 4\xi(0.75\theta_i) + 2\xi(0.5\theta_i) + 4\xi(0{,}25\theta_i) + 2.5], \tag{11}$$

where $\xi(0) = 2.5$. A comparison with the exact equation (9) showed (see Table 1) that the greatest relative error does not exceed 14 and 10.6%, respectively, in calculations by Eqs. (10) and (11); in the range of $0.15 < \theta < 0.825$ the error did not exceed 4.75 and 2.92%, respectively. The accuracy of the calculation can be increased if one takes a larger number of points of division of the interval of integration.

II. The Method of Inflection Points. After the establishment of the intervals of monotonicity of the unknown parameter (by the method of qualitative analysis [5-6] or by other means) the experimental conditions are chosen so that the graph $\theta(\xi)$ of the temperature field has an inflection point. This can occur in the two cases

$$\Lambda'(\theta) > 0, \quad \theta'(\xi) < 0; \quad \Lambda'(\theta) < 0, \quad \theta'(\xi) > 0 \tag{12}$$

(in the latter case instead of (7) we will have $\theta(0) = 0$, $\theta(\infty) = 1$, and $\theta'(\infty) = 0$). At the inflection point, in accordance with [5-6], we obtain the relation

$$\Lambda'[\theta(\xi_0)] = \frac{-2\xi_0}{\theta'(\xi_0)}, \tag{13}$$

which gives the tangent of the slope angle of the unknown parameter. It can be used to approximate the unknown parameter by linear

$$\Lambda_*(\theta) = \Lambda_*(0) + [\Lambda_*'(1) - \Lambda_*(0)]\,\theta \tag{14}$$

or quadratic

$$\Lambda_*(\theta) = K\theta^2 + L\theta + P \tag{15}$$

functions (or some other kind), where

$$P = \Lambda_*(0);\quad L = \Lambda_*(1) - \Lambda_*(0) - K;\quad K = \frac{\Lambda_*(1) - \Lambda_*(0) - \Lambda_*'[\theta(\xi_0)]}{1 - 2\theta(\xi_0)};\quad \theta(\xi_0) \neq \frac{1}{2}. \tag{16}$$

Example. Let us find $\lambda(\theta)$ determined in accordance with Eq. (9). Substituting the values of $\theta(1) = 0.38$, $\theta'(1) = -0.78$, and $\xi_0 = 1$ found from the graph into (13), (15), and (16), we obtain $\lambda_*(\theta) = 5.09\theta^2 - 1.09\theta + 1$. A comparison with (9) at the points 0.25, 0.5, and 0.75 gives relative errors of 19, 5.6, and 17.8%, respectively.

III. The Method of "Model" Solutions. As "model" solutions we understand those solutions of (1) which are easily determined and allow one to give upper and lower estimates both for the unknown temperature fields (direct problem) and for the thermophysical parameters (inverse problem). The idea of the method consists in obtaining the inequalities

$$\theta_{M1}(\xi) \leqslant \theta(\xi) \leqslant \theta_{M2}(\xi), \tag{17}$$

$$\Lambda_{M1}(\theta_{M1}) \leqslant \Lambda(\theta) \leqslant \Lambda_{M2}(\theta_{M2});\quad \Lambda_{M1}(\theta_{M1}) \geqslant \Lambda(\theta) \geqslant \Lambda_{M2}(\theta_{M2}), \tag{18}$$

which permit one to give estimates simultaneously when the unknown functions are replaced by approximate functions. Let us present the scheme of application of the method (for want of space). We will show in which cases the inequalities (17) and (18) occur. In the case of self-similar solutions let there be "model" solutions of the equation in the form

$$\varphi_{Mi}(\theta_{Mi}';\ \theta_{Mi};\ \xi) = \theta_{Mi}''\quad (i = 1, 2) \tag{19}$$

satisfying the boundary conditions (7). From (6) we subtract the expression (19) $(\Phi(\theta) \equiv 1)$* with $i = 2$:

$$\Delta\theta'' \equiv (\theta - \theta_{M2})'' = -\theta'\left[\frac{2\xi}{\Lambda(\theta)} + \frac{\Lambda'(\theta)\,\theta'}{\Lambda(\theta)}\right] - \varphi_{M2}(\theta_{M2}';\ \theta_{M2};\ \xi). \tag{20}$$

It is required to show that $\Delta\theta \le 0$, i.e., $\theta \le \theta_{M2}$. Let us assume the opposite, that $\Delta\theta > 0$, i.e., $\theta > \theta_{M2}$. Then, with allowance for the boundary conditions, we find the point $\xi = \bar{\xi}$ at which $\Delta\theta'(\bar{\xi}) = 0$ and $\theta''(\bar{\xi}) \le 0$ (i.e., $\Delta\theta$ has a maximum at the point $\xi = \bar{\xi}$). If we show that $\Delta\theta'' > 0$ then by this we also show the absence of a maximum and the validity of $\theta \le \theta_{M2}$. And the lower estimate $\theta_{M1} \le \theta$ is obtained analogously. In the course of the proof the conditions under which the inequalities (18) are valid are determined. In obtaining the "model" solutions several forms of the functions φ_{Mi} were considered:

$$\varphi_{Mi} = -\theta_{Mi}'(\xi)\,[2\xi b_i + \Lambda'(\theta_{Mi})\,\Lambda^{-1}(\theta_{Mi})\,\theta_{Mi}'], \tag{21}$$

$$\varphi_{Mi} = -\theta_{Mi}'(\xi)\,[2\xi + \Lambda'(\theta_{Mi})\,\theta_{Mi}'], \tag{22}$$

$$\varphi_{Mi} = -\theta_{Mi}'(\xi)\,[2\xi - \Lambda'(\theta_{Mi})\,\Lambda^{-1}(\theta_{Mi})\,\theta_{Mi}']. \tag{23}$$

Thus, the "model" solutions for (21) with the conditions $\theta(0) = 0$, $\theta(\infty) = 1$, $\theta'(\infty) = 0$, and (7) are written

$$\left(\int_1^{\theta_{Mi}} \Lambda(\theta)\,d\theta\right)\left(\int_1^{0} \Lambda(\theta)\,d\theta\right)^{-1} = \operatorname{erfc}\sqrt{b_i}\,\xi,\quad \theta_{Mi}' > 0,$$

$$\left(\int_1^{\theta_{Mi}} \Lambda(\theta)\,d\theta\right)\left(\int_1^{0} \Lambda(\theta)\,d\theta\right)^{-1} = \operatorname{erf}\sqrt{b_i}\,\xi,\quad \theta_{Mi}' < 0.$$

* This condition is easily obtained by an elementary change of variables.

Even simpler model solutions are obtained with

$$\varphi_{Mi} = (B_i - 2\xi)\,\theta'_{Mi}, \quad \varphi_M = \pi\xi(\theta'_M)^2 \tag{24}$$

in the form

$$\theta_{Mi} = \frac{\operatorname{erfc}(\xi - 0.5\,B_i)}{\operatorname{erfc}(-0.5\,B_i)}, \quad \theta_M = \frac{2}{\pi}\operatorname{arctg}\xi. \tag{25}$$

If one is unable to find "model" solutions for the experimental values of the field obtained (or rather, if those available give a large error) then one must substitute a linear, parabolic, or some other approximating function of the unknown parameter into (21)-(23) in order to be sure of obtaining the required degree of accuracy using the inequalities (17) and (18). Sometimes it is expedient to combine "model" solutions, taking solutions with different φ_{Mi} for the upper and lower estimates.

Example. Let $\theta' > 0$; i.e., the conditions cited above are valid. If the approximating function is taken in the form (14) with $\Lambda(0) = 0.25$ and $\Lambda(1) = 1$ then for the case (23) we obtain the solution in the form $\theta_{M2} = \frac{1}{3}(-1 + \sqrt{1 + 15\operatorname{erf}\xi})$. As the lower estimate we take the solutions (24) and (25). The maximum error reaches 28.3% at $\xi = 0$.

4. Steady Solutions. By analogy with Sec. 3, all the enumerated methods are also effective in the case of steady solutions of (1), which we rewrite as

$$\theta''_r(X) = -\theta'_r(X)X^{-1}\Lambda^{-1}(\theta_r)[\Gamma\Lambda(\theta_r) + X\Lambda'(\theta_r)\theta'_r] - \Lambda^{-1}(\theta_r)F_*(X,\ \theta_r), \tag{26}$$

where

$$\theta_r = \frac{T_r - T_{2r}}{T_{1r} - T_{2r}}, \quad X = \frac{x}{R}, \quad \Lambda(\theta_r) = \frac{\Lambda(T_r)}{\Lambda_0},$$
$$F_* = \frac{F}{F_0}, \quad F_0 = \frac{\Lambda_0(T_{1r} - T_{2r})}{R^2}. \tag{27}$$

The boundary conditions are

$$\theta_r(b) = 1, \quad \theta_r(c) = 0, \quad \theta'_r < 0 \quad (c > b) \tag{28}$$

or

$$\theta_r(b) = 0, \quad \theta_r(c) = 1, \quad \theta'_r > 0, \quad \theta_r = (T_r - T_{1r})(T_{2r} - T_{1r})^{-1}. \tag{29}$$

I. The Method of Direct Integration. In each interval of monotonicity of the coefficient of thermal conductivity we have the calculating equation [by integrating (26)]

$$\Lambda(\theta_r) = X^{-1}[\theta']^{-1}\left\{\Lambda(\theta_*)\cdot\theta'(X_*)X_*^{\Gamma} - \int_{\theta_*}^{\theta} F_*(\theta,\ X)X^{\Gamma}\,dX\right\}. \tag{30}$$

II. The Method of Inflection Points. From (26) (with $F_* \equiv 0$) it follows that inflection points are possible for

$$\Lambda'(\theta_r) > 0, \quad \theta'_r < 0; \quad \Lambda'(\theta_r) < 0, \quad \theta'_r > 0. \tag{31}$$

The condition (31) is only necessary but not sufficient for the presence of inflection points. In the presence of the inflection point $[X_0;\ \theta(X_0)]$ we have the relation

$$\Gamma\Lambda[\theta_r(X_0)] + X_0\Lambda'[\theta_r(X_0)]\cdot\theta'_r(X_0) = 0,$$

analogous to (13). Approximating functions can also be sought in the form of (14)-(16).

III. The Method of "Model" Solutions. We note that inequalities of the type of (17) and (18) are also analyzed for steady solutions (only ξ is replaced by X) and the proofs are carried out analogously. For example, one can take the appropriate linear dependence of the parameter on the temperature as the "model" solutions:

$$\Lambda(\theta) = \Lambda(0)\left[1 + \frac{1-\beta}{\beta}\theta\right]; \quad \theta = \frac{-\beta + [1-(x-b)(c-b)^{-1}(1-\beta^2)]^{\frac{1}{2}}}{1-\beta}, \quad \beta = \Lambda(0)\Lambda^{-1}(1).$$

In the case of concavity of the graph of the unknown parameter

$$\Lambda(\theta) = \Lambda(0)\{1 + [-1 + \beta^{\frac{1}{n}}]\theta\}^{-n};$$

$$\theta = \frac{1 - \left\{\beta^{\frac{1-n}{n}} - \left(\frac{x-b}{c-b}\right)[1 - \beta^{\frac{1-n}{n}}]\right\}^{\frac{1}{1-n}}}{1 - \beta^{\frac{1}{n}}};$$

in the case of convexity

$$\Lambda(\theta) = \Lambda(0)\{1 + [-1 + \beta^{-n}]\theta\}^{\frac{1}{n}};$$

$$\theta = \frac{-\beta^n + \left\{1 - \left(\frac{x-b}{c-b}\right)[1 - \beta^{n+1}]\right\}^{\frac{n}{n+1}}}{1 - \beta^n}.$$

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CONSTRUCTION OF A REGULARIZED SOLUTION TO ONE INVERSE HEAT-CONDUCTION PROBLEM WITH RANDOM ERRORS IN THE INITIAL DATA

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An analysis is made of the statistical criterion for the choice of a regulation parameter in the reconstruction of the heat flux at the surface of a body from the temperature inside the body measured with a random error.

1. The determination of the heat flux at the surface of a body from the temperature field measured inside the body is a very common inverse boundary problem of heat conduction in the analysis of experimental results [1].

Let us consider an infinite plate with a thickness d which is thermally insulated on one side. The temperature field $t(x, \tau)$ at a depth x produced by a variable heat flux $g(\tau)$ entering through the boundary $x = d$ is determined by the integral equation [2]

$$\int_0^\tau h(x, d; \tau, \tau_1) g(\tau_1) d\tau_1 = t(x, \tau), \tag{1}$$

where $h(x, d; \tau, \tau_1)$ is the Green function for a plate of finite thickness. In the case of a nonzero initial temperature distribution $t(\xi, 0)$ the right side of (1) is written in the form [2]

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